General solution of a second order non-homogenous linear difference equation with noncommutative coefficients

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SUMMARY

The detailed construction of the general solution of a second order non-homogenous linear operator-difference equation is presented. The wide applicability of such an equation as well as the usefulness of its resolutive formula is shown by studying some applications belonging to different mathematical contexts.

Keywords: difference equation, companion matrix, generating functions, noncommutativity.

1 INTRODUCTION

In this paper we report the explicit representation of the general solution of the second order non-homogenous linear operator-difference equation

$$Y_{p+2} = \mathcal{L}_0 Y_p + \mathcal{L}_1 Y_{p+1} + \phi_{p+1}, \tag{1}$$

where the unknown $\{Y_p\}_{p\in\mathbb{N}}$ as well as the non-homogenous term $\{\phi_p\}_{p\in\mathbb{N}}$ are sequences from a vectorial space V, and the coefficients $\mathcal{L}_0, \mathcal{L}_1$, are linear noncommutative operators mapping V on itself, independent from the discrete variable $p \in \mathbb{N}$. This equation encompasses interesting problems arising in very different scenarios. If, for instance, the reference space V is the complex Euclidean space $\mathbb{C}^n$, that is Y_p and ϕ_{p+1} are n-dimensional vectors, $\mathcal{L}_0$ and $\mathcal{L}_1$ $n \times n$ complex matrices, then eq. (1) is the vectorial representation of a system of second-order linear non-homogenous difference equations. As another example, let's identify V as the vectorial space of all linear operators defined on a given Hilbert space. Now, the operators $\mathcal{L}_0$ and $\mathcal{L}_1$ act upon operators and for this reason are called superoperators. The master equations appearing in the theory of open quantum systems provide examples of equations belonging to this class [5]. It is of relevance to emphasize from the very beginning that the ingredients Y_p, ϕ_p, $\mathcal{L}_0$ and $\mathcal{L}_1$ of eq. (1) may be also interpreted as elements of an assigned algebra V. Let's consider, for example, V as the

noncommutative algebra of all square matrices of order n, that is $M_n[\mathbb{C}]$. Then, eq. (1) defines a second order non-homogenous linear matrix-difference equation, where $Y_p, \phi_p, \mathfrak{L}_0$ and $\mathfrak{L}_1$ belong to $M_n(\mathbb{C})$. We wish further emphasis that if V is the vectorial space of the smooth functions over an interval I, that is $C^{\infty}(I)$, then eq. (1) represents a wide class of functional-difference equations [1], including difference-differential equations or integro-difference equations [2–4]. These few examples motivate the interest toward the search of techniques for solving the operator eq.(1), with $\mathfrak{L}_0, \mathfrak{L}_1$ noncommutative coefficients.

In this paper we cope with such a problem and succeed in giving its explicit solution leaving unspecified the abstract "support space" wherein eq. (1) is formulated. This means that we do not choose from the very beginning the mathematical nature of its ingredients, rather we only require that all the symbols and operations appearing in eq. (1) are meaningful. Accordingly, "vectors" Y_p may be added, this operation being commutative and, at the same time, may be acted upon by $\mathfrak{L}_0$ or $\mathfrak{L}_1$ (hereafter called operators) transforming themselves into other "vectors"of V. The symbol $Y_0 = 0$ simply denotes, as usual, the neutral element of the underlying space. Finally we put $(\mathfrak{L}_a\mathfrak{L}_b)Y \equiv \mathfrak{L}_a(\mathfrak{L}_b Y) \equiv \mathfrak{L}_a\mathfrak{L}_b Y$ with a or $b = 0, 1$ and define addition between operators through linearity.

The paper is organized as follows.
The first section presents the solution of an arbitrary Cauchy problem associated with eq. (1). Some interesting consequences of such a result are derived in the subsequent section. The practical usefulness of our resolutive formula is shown in the third section where we solve some nontrivial functional-difference and integral-difference equations. Some concluding remarks are presented in the last section.

2 EXPLICIT CONSTRUCTION OF THE RESOLUTIVE FORMULA OF EQ.(1)

Let's begin by recalling that if $\{Y_p^*\}_{p\in\mathbb{N}}$ and $\{Y_p\}_{p\in\mathbb{N}}$ are solutions of eq.(1), then $\{Y_p^H\}_{p\in\mathbb{N}}$ defined as $Y_p - Y_p^* \equiv Y_p^H$ is a solution of the associated homogenous equation

$$Y_{p+2} = \mathfrak{L}_0 Y_p + \mathfrak{L}_1 Y_{p+1} \tag{2}$$

Thus, as for the linear differential equations, and independently from the noncommutative nature of $\mathfrak{L}_0$ and $\mathfrak{L}_1$, solving eq. (1) amounts at being able to construct the general integral of eq. (2) and to find out a particular solution of eq. (1). To this end, we start with the following theorem which extends a recently published result [7] concerning the exact resolution of the following Cauchy problem

$$\begin{cases} Y_{p+2} = \mathfrak{L}_0 Y_p + \mathfrak{L}_1 Y_{p+1}, \\ Y_0 = 0, \quad Y_1 = B \end{cases} . \tag{3}$$

Theorem 1. *The solution of the Cauchy problem*

$$\begin{cases} Y_{p+2} = \mathfrak{L}_0 Y_p + \mathfrak{L}_1 Y_{p+1} \\ Y_0 = A, \quad Y_1 = B \end{cases} , \tag{4}$$

can be written as

$$Y_p^{(H)} = \alpha_p A + \beta_p B, \tag{5}$$

where the operators α_p *and* β_p *have the following form*

$$\alpha_p = \begin{cases} \sum_{t=0}^{[\frac{p-2}{2}]} \{\mathcal{L}_0^{(t)} \mathcal{L}_1^{(p-2-2t)}\} \mathcal{L}_0 & if \quad p \geq 2 \\ 0 & if \quad p = 1 \\ E & if \quad p = 0 \end{cases}, \tag{6}$$

$$\beta_p = \begin{cases} \sum_{t=0}^{[\frac{p-1}{2}]} \{\mathcal{L}_0^{(t)} \mathcal{L}_1^{(p-1-2t)}\} & if \quad p \geq 2 \\ E & if \quad p = 1 \\ 0 & if \quad p = 0 \end{cases}, \tag{7}$$

We recall that the mathematical symbol $\{\mathcal{L}_0^{(u)} \mathcal{L}_1^{(v)}\}$, in accordance with ref [7], denotes the sum of all possible distinct permutations of u factors $\mathcal{L}_0$ and v factors $\mathcal{L}_1$, while 0, $E : V \to V$ define the null and the identity operator in V, respectively. We omit the proof of this theorem since it is practically coincident with that given in ref [7]. Here instead we demonstrate the following

Theorem 2. *Eq. (1) admits the particular solution*

$$Y_p^* = \begin{cases} \sum_{r=1}^{p-1} \beta_{p-r} \phi_r, & if \quad p \geq 2 \\ 0, & if \quad p = 0, 1 \end{cases} \tag{8}$$

Proof. It is immediate to verify, by direct substitution, that the sequence given by eq. (8) satisfies eq. (1) written for $p = 0$ and $p = 1$. To this end, it is enough to exploit eqs. (7) and (8) getting $Y_2^* = \beta_1 \phi_1 = \phi_1$ and $Y_3^* = \beta_2 \phi_1 + \beta_1 \phi_2 = \mathcal{L}_1 \phi_1 + \phi_2$.
For a generic $p \geq 2$, introducing Y_p^* in the right hand of eq. (1) yields

$$\begin{aligned} & \mathcal{L}_0 \sum_{r=1}^{p-1} \beta_{p-r} \phi_r + \mathcal{L}_1 \sum_{r=1}^{p} \beta_{p+1-r} \phi_r + \phi_{p+1} \\ = & \sum_{r=1}^{p-1} (\mathcal{L}_0 \beta_{p-r} + \mathcal{L}_1 \beta_{p+1-r}) \phi_r + \mathcal{L}_1 \beta_1 \phi_p + \phi_{p+1} \end{aligned} \tag{9}$$

Applying theorem (1) to the Cauchy problem expressed by eq. (3), we easily deduce that for $p \geq 2$ and $r = 1, 2, \ldots, p-1$ the following operator identity

$$\mathcal{L}_0 \beta_{p-r} + \mathcal{L}_1 \beta_{p+1-r} = \beta_{p+2-r}, \tag{10}$$

holds. Thus, the expression given by eq. (9) may be cast as follows

$$\sum_{r=1}^{p-1} \beta_{p+2-r} \phi_r + \beta_{p+2-(p)} \phi_p + \beta_{p+2-(p+1)} \phi_{p+1} = \sum_{r=1}^{p+1} \beta_{p+2-r} \phi_r \tag{11}$$

where we have exploited the identity $\beta_2 = \mathcal{L}_1 \beta_1$ based on eq. (7). Since the right hand of eq. (11) coincides with Y_{p+2}^* as given by eq. (8), we may conclude that $\{Y_p^*\}_{p \in \mathbb{N}}$, expressed by eq. (8), provides a particular solution of eq. (1). $\square$

On the basis of theorem (1) and (2) we hence may state our main result, that is

Theorem 3. *The solution of the Cauchy problem*

$$\left\{ \begin{array}{r} Y_{p+2} = \mathcal{L}_0 Y_p + \mathcal{L}_1 Y_{p+1} + \phi_{p+1} \\ Y_0 = A, \quad Y_1 = B \end{array} \right. , \tag{12}$$

is

$$Y_p = \alpha_p A + \beta_p B + \sum_{r=1}^{p-1} \beta_{p-r} \phi_r \tag{13}$$

where A and B are generic admissible initial conditions and α_p and β_p are defined by eqs. (6) *and* (7)*, respectively.*

We emphasize that eq. (13) furnishes a recipe to solve explicitly, that is in terms of its ingredients $\mathcal{L}_0, \mathcal{L}_1$ and $\{\phi_{p+1}\}_{p\in\mathbb{N}}$, the general Cauchy problem expressed by eq. (12). In the subsequent sections we will highlight that our result is effectively exploitable, providing indeed a useful approach to solve problems belonging to very different mathematical contexts. This circumstance adds a further robust motivation to investigate eq. (1) and its consequences.

We conclude this section looking for the structural form assumed by eq. (13) solely relaxing the noncommutativity between the two operator coefficients $\mathcal{L}_0$ and $\mathcal{L}_1$. To this end, it is useful to recall the definition of the Chebyshev polynomials of the second kind $\mathcal{U}_p(x), x \in \mathbb{C}$ [8]

$$\mathcal{U}_p(x) = \sum_{m=0}^{[p/2]} (-1)^m \frac{(p-m)!}{m!(p-2m)!} (2x)^{p-2m} \tag{14}$$

Indeed, taking into consideration that the number of all the different terms appearing in the operator symbol $\{\mathcal{L}_0^{(u)}\mathcal{L}_1^{(v)}\}$ coincides with the binomial coefficient $\left(\begin{array}{c} u+v \\ m \end{array} \right)$, with $m = min(u,v)$ as well as assuming the existence of the operator $(-\mathcal{L}_0)^{-\frac{1}{2}}$, then the operators α_p and β_p for $p \geq 2$ may be cast as follows

$$\alpha_p = -(-\mathcal{L}_0)^{\frac{p}{2}} \mathcal{U}_{p-2} \left(\frac{1}{2} \mathcal{L}_1 (-\mathcal{L}_0)^{-\frac{1}{2}} \right) \tag{15}$$

and

$$\beta_p = (-\mathcal{L}_0)^{\frac{p-1}{2}} \mathcal{U}_{p-1} \left(\frac{1}{2} \mathcal{L}_1 (-\mathcal{L}_0)^{-\frac{1}{2}} \right) \tag{16}$$

where $\mathcal{U}_p \left(\frac{1}{2} \mathcal{L}_1 (-\mathcal{L}_0)^{-\frac{1}{2}} \right)$ means the operator value of the polynomial $\mathcal{U}_p$ defined in accordance with eq.(14) for $x = \left(\frac{1}{2} \mathcal{L}_1 (-\mathcal{L}_0)^{-\frac{1}{2}} \right)$. Thus, the solution of Cauchy problem (12) may be rewritten, for $p \geq 2$, as

$$\begin{aligned} Y_p = (-\mathcal{L}_0)^{\frac{p-1}{2}} \mathcal{U}_{p-1} \left(\frac{1}{2} \mathcal{L}_1 (-\mathcal{L}_0)^{-\frac{1}{2}} \right) B - (-\mathcal{L}_0)^{\frac{p}{2}} \mathcal{U}_{p-2} \left(\frac{1}{2} \mathcal{L}_1 (-\mathcal{L}_0)^{-\frac{1}{2}} \right) A + \\ + \sum_{r=1}^{p-1} (-\mathcal{L}_0)^{\frac{p-r-1}{2}} \mathcal{U}_{p-r-1} \left(\frac{1}{2} \mathcal{L}_1 (-\mathcal{L}_0)^{-\frac{1}{2}} \right) \phi_r \end{aligned} \tag{17}$$

where $Y_0 = A$ and $Y_1 = B$ are the prescribed initial conditions.

3 SOME CONSEQUENCES OF THE RESOLUTIVE FORMULA

The mathematical literature offers several ways of solving linear second difference equations such as the matrix method or the generating function method. In the following we will heuristically generalized these methods to the operator case. The novelty of our method enables to deduce, by comparison with these approaches, some interesting consequent identities. Indeed, the second-order operator difference equation (1) may be traced back to the first-order vectorial representation

$$\mathbf{Y}_{p+1} = C_1\mathbf{Y}_p + \Phi_{p+1} \tag{18}$$

where $\mathbf{Y}_p = \begin{pmatrix} Y_p \\ Y_{p+1} \end{pmatrix}$, $C_1 = \begin{pmatrix} 0 & E \\ \mathcal{L}_0 & \mathcal{L}_1 \end{pmatrix}$, $\Phi_{p+1} = \begin{pmatrix} 0 \\ \phi_{p+1} \end{pmatrix}$, $\mathbf{Y}_0 = \begin{pmatrix} A \\ B \end{pmatrix}$.

Successive iterations easily lead us to the formal solution

$$\mathbf{Y}_p = C_1^p\mathbf{Y}_0 + \sum_{r=1}^{p} C_1^{p-r}\Phi_r \tag{19}$$

On this basis, the solution of eq.(1) may be written as [11]

$$Y_p = P_1C_1^p\mathbf{Y}_0 + P_1\sum_{r=1}^{p} C_1^{p-r}R_1\Phi_r \tag{20}$$

where $P_1 = (E \quad 0)$ and $R_1 = \begin{pmatrix} 0 \\ E \end{pmatrix}$. This solution is of practical use only if we are able to evaluate the general integer power of the companion matrix C_1. Exploiting our procedure of writing the solution of eq. (1), the vector $\mathbf{Y}_p$ may be expressed, accordingly with eq. (13), in terms of operator sequences α_p and β_p like

$$\mathbf{Y}_p = \begin{pmatrix} \alpha_pA + \beta_pB + \sum\limits_{r=1}^{p-1}\beta_{p-r}\phi_r \\ \alpha_{p+1}A + \beta_{p+1}B + \sum\limits_{r=1}^{p}\beta_{p+1-r}\phi_r \end{pmatrix} = \begin{pmatrix} \alpha_p & \beta_p \\ \alpha_{p+1} & \beta_{p+1} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} + \begin{pmatrix} \sum\limits_{r=1}^{p-1}\beta_{p-r}\phi_r \\ \sum\limits_{r=1}^{p}\beta_{p+1-r}\phi_r \end{pmatrix} \tag{21}$$

Confining ourselves to the homogenous version of eq. (1), that is putting $\phi_{p+1} = 0$ into eqs. (19) and (21), we get the formula for the p-th power of the companion matrix C_1 as follows

$$C_1^p = \begin{pmatrix} 0 & E \\ \mathcal{L}_0 & \mathcal{L}_1 \end{pmatrix}^p = \begin{pmatrix} \alpha_p & \beta_p \\ \alpha_{p+1} & \beta_{p+1} \end{pmatrix} \tag{22}$$

Another possible way of treating eq.(1) is via the generating functions method [9,10]. We recall that, given the sequence $\{Y_p\}_{p\in\mathbb{N}}$, the associated generating function $Y(s), s \in \mathbb{C}$ is defined as

$$\mathcal{G}Y_p \equiv Y(s) \equiv \sum_{p=0}^{\infty} Y_p s^p \tag{23}$$

under the assumption that the series converges when $|s| \leq \xi$, for some positive number ξ. The advantage of this method consists in the systematical possibility of transforming a difference equation in an algebraic one in the unknown $Y(s)$. In order to apply such approach to the operator-difference equation given by (1), we stipulate that $\mathcal{G}[\mathcal{L}_iY_p]$, $\mathcal{L}_i(\mathcal{G}Y_p)$ are both defined and that

$\mathcal{G}[\mathcal{L}_i Y_p] = \mathcal{L}_i(\mathcal{G}Y_p), i = 0, 1$. Accordingly, heuristically, we transform both sides of eq. (1) getting

$$\frac{Y(s) - A - Bs}{s^2} = \mathcal{L}_1 \frac{Y(s) - A}{s} + \mathcal{L}_0 Y(s) + \Phi(s)$$

Thus, assuming the existence of $(E - \mathcal{L}_1 s - \mathcal{L}_0 s^2)^{-1}$ within the convergence disk, we have

$$Y(s) = (E - \mathcal{L}_1 s - \mathcal{L}_0 s^2)^{-1} A + (E - \mathcal{L}_1 s - \mathcal{L}_0 s^2)^{-1}(B - \mathcal{L}_1 A)s + (E - \mathcal{L}_1 s - \mathcal{L}_0 s^2)^{-1}\Phi(s)s^2 \quad (24)$$

or equivalently

$$Y(s) = (E - \mathcal{L}_1 s - \mathcal{L}_0 s^2)^{-1}[E - \mathcal{L}_1 s]A + (E - \mathcal{L}_1 s - \mathcal{L}_0 s^2)^{-1}Bs + (E - \mathcal{L}_1 s - \mathcal{L}_0 s^2)^{-1}\Phi(s)s^2 \quad (25)$$

On the other hand, accordingly with eqs. (23) and (13) it holds that

$$Y(s) = \sum_{p=0}^{\infty}(\alpha_p A + \beta_p B + \sum_{r=1}^{p-1}\beta_{p-r}\phi_r)s^p \quad (26)$$

Thus, one notes that imposing $\phi_{p+1} \equiv 0$ and $B = 0$, respectively $A = 0$, we heuristically find the generating function of the operator sequences α_p and β_p in the closed form as

$$\mathcal{G}\alpha_p \equiv \sum_{p=0}^{\infty}\alpha_p s^p := (E - \mathcal{L}_1 s - \mathcal{L}_0 s^2)^{-1}[E - \mathcal{L}_1 s] \quad (27)$$

respectively

$$\mathcal{G}\beta_p \equiv \sum_{p=0}^{\infty}\beta_p s^p := (E - \mathcal{L}_1 s - \mathcal{L}_0 s^2)^{-1}s \quad (28)$$

The particular case $\mathcal{L}_0 = -E$ reproduces the generating functions of the Chebyshev polynomials of second kind. Extracting indeed for the sake of convenience the first two terms of the series, that is writing $\sum_{p=0}^{\infty}\alpha_p s^p = \alpha_0 + \alpha_1 s + \sum_{p=2}^{\infty}\alpha_p s^p$ with the help of eq. (15) and $\alpha_0 = E, \alpha_1 = 0$ we get

$$(E - \mathcal{L}_1 s + s^2)^{-1}[E - \mathcal{L}_1 s] = E - \sum_{p=2}^{\infty}\mathcal{U}_{p-2}[\frac{\mathcal{L}_1}{2}]s^p \quad (29)$$

The eq. (29) easily determines the generating function of the sequence $\{U_p[\mathcal{L}_1/2]\}_{p\in\mathbb{N}}$ in the form

$$\sum_{p=2}^{\infty}\mathcal{U}_{p-2}[\frac{\mathcal{L}_1}{2}]s^p = (E - \mathcal{L}_1 s + s^2)^{-1}[E - \mathcal{L}_1 s + s^2 - E + \mathcal{L}_1 s]$$

$$\Leftrightarrow \sum_{p=0}^{\infty}\mathcal{U}_p[\frac{\mathcal{L}_1}{2}]s^p = (E - \mathcal{L}_1 s + s^2)^{-1}E \quad (30)$$

The novel results obtained in this paper exploiting our resolutive formula (eqs. (22), (27), (28)) clearly evidence that our recipe to manage eq.(1) successfully integrate with other resolutive methods. Thus, we may claim that our resolutive formula do not possess a formal character only, since it helps to provide new interesting identities.

4 APPLICATIONS OF OUR RESOLUTIVE FORMULA

An example of matrix-difference equation coped with our formula

Let us consider the second order matrix-difference eqution

$$Y_{p+2} = M_0 Y_p + M_1 Y_{p+1} + \Phi_{p+1} \tag{31}$$

where $M_0, M_1 \in \mathcal{M}_n(\mathbb{C})$ are noncommutative nilpotent matrices of index 2, that is $M_i^2 = 0, \quad i = 0, 1$. Prescribing the initial conditions $Y_0 = A, Y_1 = B$, then the solution of this equation is given by the eq. (13). The analysis of the matrix term $\{M_0^{(u)} M_1^{(v)}\}$ which appears in the composition of the matrix-operator α_p and β_p brings to light interesting peculiarities due to the specific nature of the coefficients M_0 and M_1. By definition, the term $\{M_0^{(u)} M_1^{(v)}\}$ represents the sum of all possible terms of u factor M_0 and v factors M_1. Thus, it is quite simple to deduce that now the matrix-term of the form $M_0^{\nu_1} M_1^{\nu_2} M_0^{\nu_3} M_1^{\nu_4} \ldots$ is equal with zero, if $\nu_i > 1, (\forall) i$. Hence, we deduce that the operator $\{M_0^{(u)} M_1^{(v)}\}$ survives solely when $u = v$ or $v = u \pm 1$. Indeed, when $u = v, u \geq 2$, then in the sum $\{M_0^{(u)} M_1^{(u)}\}$ survive only the terms $\underbrace{[M_0 M_1][M_0 M_1] \ldots [M_0 M_1]}_{u-times}$ and $\underbrace{[M_1 M_0][M_1 M_0] \ldots [M_1 M_0]}_{u-times}$. Further, the only nonvanishing matrix-terms $\{M_0^{(u)} M_1^{(u+1)}\}$ are $M_1 \underbrace{[M_0 M_1][M_0 M_1] \ldots [M_0 M_1]}_{u-times}$, as well as from $\{M_0^{(u)} M_1^{(u-1)}\}$ the terms $\underbrace{[M_0 M_1][M_0 M_1] \ldots [M_0 M_1]}_{(u-1)-times} M_0$, respectively. Exploiting the above results for the matrix term $\{M_0^{(t)} M_1^{(p-1-2t)}\}$ we establish that

$$\beta_p = \begin{cases} \{M_0^{(\frac{p-1}{3})} M_1^{(\frac{p-1}{3})}\} = \underbrace{[M_0 M_1] \ldots [M_0 M_1]}_{k-times} + \underbrace{[M_1 M_0] \ldots [M_1 M_0]}_{k-times}, & p = 3k+1 \\ \{M_0^{(\frac{p-2}{3})} M_1^{(\frac{p+1}{3})}\} = M_1 \underbrace{[M_0 M_1] \ldots [M_0 M_1]}_{k-times}, & p = 3k+2 \\ \{M_0^{(\frac{p}{3})} M_1^{(\frac{p-3}{3})}\} = \underbrace{[M_0 M_1] \ldots [M_0 M_1]}_{(k-1)-times} M_0, & p = 3k, \end{cases} \tag{32}$$

where $k = 1, 2, \ldots$. Similarly, we have that

$$\alpha_p = \begin{cases} \{M_0^{(\frac{p-2}{3})} M_1^{(\frac{p-2}{3})}\} M_0 = \underbrace{[M_0 M_1] \ldots [M_0 M_1]}_{k-times} M_0, \quad p = 3k+2 \\ \{M_0^{(\frac{p-3}{3})} M_1^{(\frac{p}{3})}\} M_0 = M_1 \underbrace{[M_0 M_1] \ldots [M_0 M_1]}_{(k)-times} M_0, \quad p = 3k+3 \\ \{M_0^{(\frac{p-1}{3})} M_1^{(\frac{p-4}{3})}\} M_0 = (\underbrace{[M_0 M_1] \ldots [M_0 M_1]}_{(k-1)-times} M_0) M_0 = 0, \quad p = 3k+1 \end{cases} \tag{33}$$

Hence, we may write the solution into a closed form

$$Y_p = \left[\delta_{\frac{p-2}{3},[\frac{p-2}{3}]} (M_0 M_1)^{[\frac{p-2}{3}]} M_0 + \delta_{\frac{p-3}{3},[\frac{p-3}{3}]} M_1 (M_0 M_1)^{[\frac{p-3}{3}]} M_0\right] A +$$

$$[\delta_{\frac{p-1}{3},[\frac{p-1}{3}]} (M_0 M_1)^{[\frac{p-1}{3}]} + \delta_{\frac{p-1}{3},[\frac{p-1}{3}]} (M_1 M_0)^{[\frac{p-1}{3}]} + \delta_{\frac{p-2}{3},[\frac{p-2}{3}]} M_1 (M_0 M_1)^{[\frac{p-2}{3}]} + \delta_{\frac{p-3}{3},[\frac{p-3}{3}]} (M_0 M_1)^{[\frac{p-3}{3}]} M_0] B$$

$$+ \sum_{r=1}^{p-1} [\delta_{\frac{p-r-1}{3},[\frac{p-r-1}{3}]} \left((M_0 M_1)^{[\frac{p-r-1}{3}]} + (M_1 M_0)^{[\frac{p-r-1}{3}]} \right) +$$
$$\delta_{\frac{p-r-2}{3},[\frac{p-r-2}{3}]} M_1 (M_0 M_1)^{[\frac{p-r-2}{3}]} + \delta_{\frac{p-r-3}{3},[\frac{p-r-3}{3}]} (M_0 M_1)^{[\frac{p-r-3}{3}]} M_0] \Phi_r \tag{34}$$

An example of functional-difference equation coped with our method

The three-term recurrence relation

$$f_{p+2}(t) = -f_p(t - \tau_0) + f_{p+1}(t + \tau_1) \tag{35}$$

with the initial conditions $A = f_0(t)$ and $B = f_1(t)$ is an example of functional difference equation, traceable back to eq. (1). It is indeed well-known that if $f(t)$ is a function of class C^∞, then the translation of its independent variable from t to $t + \tau$ can be represented as the effect on the same function of the operator $exp[\tau \frac{d\cdot}{dt}] = \sum_{k=0}^{\infty} \frac{1}{k!} [\tau \frac{d\cdot}{dt}]^k$. This operator appears in a natural way when one studies problems characterized by translational invariance in a physical context [12]. Thus, by putting $\mathfrak{L}_i = (-1)^{i+1} exp[(-1)^{i+1} \tau_i \frac{d.}{dt}], i = 0, 1$ the commutativity property of the two operator coefficients $\mathfrak{L}_0$ and $\mathfrak{L}_1$ allows us to write down the solution of eq. (35) as follows

$$f_p(t) = \exp[-(\frac{p-1}{2}) \tau_0 \frac{d}{dt}] \mathfrak{U}_{p-1} \left[\frac{1}{2} \exp[(\tau_1 + \frac{\tau_0}{2}) \frac{d}{dt}]\right] f_1(t) -$$
$$\exp[-(\frac{p}{2}) \tau_0 \frac{d}{dt}] \mathfrak{U}_{p-2} \left[\frac{1}{2} \exp[(\tau_1 + \frac{\tau_0}{2}) \frac{d}{dt}]\right] f_0(t) \tag{36}$$

Exploiting eq. (14) we may write down that

$$f_p(t) = \sum_{k=0}^{[p-1/2]} (-1)^k \binom{p-1-k}{k} f_1\left(t + (p-1-2k)\tau_1 - k\tau_0\right) - \tag{37}$$

$$\sum_{k=0}^{[p-2/2]} (-1)^k \binom{p-2-k}{k} f_0\left(t + (p-2-2k)\tau_1 - (k+1)\tau_0\right) \tag{38}$$

Imposing, for example, the following initial conditions $f_0(t) = e^{-t}$ and $f_1(t) = e^t$ we get

$$f_p(t) = \exp[-(\frac{p-1}{2})\tau_0]\mathcal{U}_{p-1}\left[\frac{1}{2}\exp[(\tau_1 + \frac{\tau_0}{2})]\right] e^t - \tag{39}$$

$$\exp[(\frac{p}{2})\tau_0]\mathcal{U}_{p-2}\left[\frac{1}{2}\exp[-(\tau_1 + \frac{\tau_0}{2})]\right] e^{-t} \tag{40}$$

An example of integro-difference equation coped with our method

Consider the difference-differential equation

$$f'_{p+2}(t) = \beta f_{p+1}(t) + \alpha f_p(t), \quad \alpha, \beta \in \mathbb{R}, \quad p = 0, 1, \ldots \tag{41}$$

where $f_p(t)$ is a $C^\infty(I)$ function with $f_0(t)$, $f_1(t)$ and $\{f_p(0), \quad p = 0, 1, \ldots\}$ prescribed functions. The above equation may be rewritten in the equivalent form

$$f_{p+2}(t) = \mathfrak{L}_1 f_{p+1}(t) + \mathfrak{L}_0 f_p(t) + f_{p+2}(0) \tag{42}$$

where $\mathfrak{L}_0 = \alpha\mathfrak{L}$, $\mathfrak{L}_1 = \beta\mathfrak{L}$ and $\mathfrak{L}(\cdot) = \int_0^t \cdot d\tau$. Eq. (42) is a particular case of eq.(1).
The explicit solution of this equation requires the knowledge of the operator terms α_p and β_p. One remarks that β_p is the sum of $\left[\frac{p-1}{2}\right] + 1$ operator terms of the form $\{\mathfrak{L}_0^t \mathfrak{L}_1^{p-1-2t}\}$. Because $\mathfrak{L}_0 = \alpha\mathfrak{L}$ and $\mathfrak{L}_1 = \beta\mathfrak{L}$ then, for a finite p, holds

$$\{\mathfrak{L}_0^k \mathfrak{L}_1^{p-1-2k}\} = \alpha^k \beta^{p-1-2k} \binom{p-1-k}{k} \mathfrak{L}^{p-1-k} \tag{43}$$

Therefore, by direct substitution into eq. (7) it follows that

$$\beta_p = \sum_{k=0}^{\left[\frac{p-1}{2}\right]} \alpha^k \beta^{p-1-2k} \binom{p-1-k}{k} \mathfrak{L}^{p-1-k} \tag{44}$$

Similarly, we have that

$$\alpha_p = \sum_{k=0}^{\left[\frac{p-2}{2}\right]} \alpha^{k+1} \beta^{p-2-2k} \binom{p-2-k}{k} \mathfrak{L}^{p-1-k} \tag{45}$$

The solution of the corresponding homogenous equation in accordance with the prescribed initial conditions is then

$$f_p^{(H)}(t) = \sum_{k=0}^{\left[\frac{p-2}{2}\right]} \alpha^{k+1}\beta^{p-2-2k} \begin{pmatrix} p-2-k \\ k \end{pmatrix} \mathfrak{L}^{p-1-k}\left(f_0(t)\right) + \sum_{k=0}^{\left[\frac{p-1}{2}\right]} \alpha^{k}\beta^{p-1-2k} \begin{pmatrix} p-1-k \\ k \end{pmatrix} \mathfrak{L}^{p-1-k}\left(f_1(t)\right) \quad (46)$$

Exploiting our central theorem (2), we may claim that

$$f_p^* = \sum_{m=1}^{p-1} \beta_{p-m} f_{m+1}(0) = \sum_{m=1}^{p-2} \beta_{p-m} f_{m+1}(0) + f_p(0) \quad (47)$$

is the particular solution of the nonhomogenous equation for which $f_0 = f_1 = 0, (\forall)t$. Equivalently, we have that

$$f_p^* = \sum_{m=1}^{p-2} \sum_{k=0}^{\left[\frac{p-m-1}{2}\right]} \alpha^{k}\beta^{p-m-1-2k} \begin{pmatrix} p-m-1-k \\ k \end{pmatrix} \mathfrak{L}^{p-m-1-k}\left(f_{m+1}(0)\right) + f_p(0) \quad (48)$$

But, as shown in the Appendix, we may prove that

$$\mathfrak{L}^n(f(t)) = \int_0^t dt_n \int_0^{t_n} dt_{n-1} \int_0^{t_{n-1}} dt_{n-2} \ldots \int_0^{t_3} dt_2 \int_0^{t_2} f(t_1)dt_1 = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)d\tau, \quad (49)$$

so that we may write down that

$$f_p^{(H)}(t) = \sum_{k=0}^{\left[\frac{p-2}{2}\right]} \alpha^{k+1}\beta^{p-2-2k} \begin{pmatrix} p-2-k \\ k \end{pmatrix} \frac{1}{(p-2-k)!}\int_0^t (t-\tau)^{p-2-k} f_0(\tau)d\tau + \sum_{k=0}^{\left[\frac{p-1}{2}\right]} \alpha^{k}\beta^{p-1-2k} \begin{pmatrix} p-1-k \\ k \end{pmatrix} \frac{1}{(p-2-k)!}\int_0^t (t-\tau)^{p-2-r} f_1(\tau)d\tau \quad (50)$$

and

$$f_p^* = \sum_{m=1}^{p-2} \sum_{k=0}^{\left[\frac{p-m-1}{2}\right]} \alpha^{k}\beta^{p-m-1-2k} \begin{pmatrix} p-m-1-k \\ k \end{pmatrix} \frac{1}{(p-m-2-k)!}\int_0^t (t-\tau)^{p-m-2-k} f_{m+1}(0)d\tau + f_p(0)$$

Hence, the general solution of the proposed integral-difference equation is

$$f_p(t) = \sum_{k=0}^{\left[\frac{p-2}{2}\right]} \alpha^{k+1}\beta^{p-2-2k} \begin{pmatrix} p-2-k \\ k \end{pmatrix} \frac{1}{(p-2-k)!} \int_0^t (t-\tau)^{p-2-k} f_0(\tau)d\tau +$$

$$\sum_{k=0}^{\left[\frac{p-1}{2}\right]} \alpha^{k}\beta^{p-1-2k} \begin{pmatrix} p-1-k \\ k \end{pmatrix} \frac{1}{(p-2-k)!} \int_0^t (t-\tau)^{p-2-k} f_1(\tau)d\tau + \quad (51)$$

$$\sum_{m=1}^{p-2} \sum_{k=0}^{\left[\frac{p-m-1}{2}\right]} \alpha^{k}\beta^{p-m-1-2k} \begin{pmatrix} p-m-1-k \\ k \end{pmatrix} \frac{t^{p-m-1-k}}{(p-m-2-k)!(p-m-1-k)!} f_{m+1}(0) + f_p(0) \quad (52)$$

5 CONCLUSIVE REMARKS

The novel and mean theoretical result of this paper is expressed by theorem (2) with which we demonstrate that eq. (8) provides a particular solution of eq. (1). This result together with theorem (1) completes the resolution of this equation enabling us to write down formula (13) for its general solution. The operator character of eq. (1) and, as a consequence, the presence of generally noncommuting coefficients is the key to understand why such an equation may represent the canonical form of equations seemingly not related each other. The consequences of eq. (13) and the applications reported in this paper, besides being interesting in their own, demonstrate indeed both the wide applicability of eq. (1) as well as the practical usefulness of its resolutive formula.

A APPENDIX

For the sake of completeness we here report a proof of the well-known follwing identity

$$\int_0^t dt_n \int_0^{t_n} dt_{n-1} \int_0^{t_{n-1}} dt_{n-2} \dots \int_0^{t_3} dt_2 \int_0^{t_2} f(t_1)dt_1 = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau)d\tau, \quad (53)$$

where $f(t)$ is a C^∞-function. The mathematical induction procedure will be exploited.
For $n=1$ the above formula becomes the identity

$$\int_0^t f(t_1)dt_1 = \int_0^t f(\tau)d\tau \quad (54)$$

Let's suppose that the formula 53 holds for any $r \leq n$ and we prove it validity for $n+1$. To proceed, it is convenient to define

$$F_r(t) = \int_0^t dt_r \int_0^{t_r} dt_{r-1} \int_0^{t_{r-1}} dt_{r-2} \dots \int_0^{t_3} dt_2 \int_0^{t_2} f(t_1)dt_1 \quad (55)$$

writing down eq.(53) as follows

$$F_n(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau)d\tau$$

It is obviously that by definition

$$F_{r+1}(t) = \int_0^t F_r(t_{r+1}) dt_{r+1}, \quad r \geq 1 \tag{56}$$

By the induction hypothesis we have that

$$F'_{n+1} = F_n(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau \tag{57}$$

Our problem becomes the resolution of the following Cauchy problem

$$\begin{cases} F'_{n+1}(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau \\ F_{n+1}(0) = 0 \end{cases} \tag{58}$$

From the well-known Leibnitz identity

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(x,y) dy = \int_{a(x)}^{b(x)} \frac{\partial f(x,y)}{\partial x} dy + b'(x) f(x, b(x)) - a'(x) f(x, a(x)) \tag{59}$$

we have that

$$\frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau = \frac{1}{n(n-1)!} \int_0^t \frac{\partial \left[(t-\tau)^n f(\tau) \right]}{\partial t} d\tau = \frac{d}{dt} \left[\frac{1}{n!} \int_0^t (t-\tau)^n f(\tau) d\tau \right] \tag{60}$$

Hence, the differential equation for $F_{n+1}(t)$ may be rewritten as

$$F'_{n+1}(t) = \frac{d}{dt} \left[\frac{1}{n!} \int_0^t (t-\tau)^n f(\tau) d\tau \right] \tag{61}$$

such that the Cauchy problem (58) has the solution

$$F_{n+1}(t) = \frac{1}{n!} \int_0^t (t-\tau)^n f(\tau) d\tau \tag{62}$$

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